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Stochastic Analysis of Van der Pol OscillatorModel Using Wiener HermiteExpansion Linked by Multi-Step Differential TransformTechnique

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Abstract
We study a model related to Van der Poloscillatorunder an external stochastic excitation described by white noise process. This study is limited to find the Gaussian behavior of the stochastic solution processes related to the model. Under the application ofWiener-Hermite expansion, a deterministic system is generated to describe the Gaussian solution parameters (Mean and Variance).The deterministic system solution is approximated by applying the multi-stepdifferential transformedmethodand the results are compared with NDSolveMathematica 10 package. Some case studies are considered to illustrate some comparisons for the obtained results related to the Gaussian behavior parameters.
Keywords:Wiener-Hermite expansion; Multi-step differential transformed method; NDSolveMathematica 10 package;Van der Pol oscillator; Deterministic systems.

I. Introduction

In recent years, the analysis of the nonlinear oscillator subjected to random excitation has been studied by many investigators. This phenomena is described by a stochastic differential equation under deterministic initial conditions and its solution behavior will be become a stochastic process due to existing an external stochastic term in the mathematical modeling. Some examples related to this phenomena simulate the vibrational studies of mechanical, electrical systems, earthquake disturbances,wind load in structural analysis, noise-corrupted signals in communication theory, and the motion of the sea or ground roughness in vehicle dynamics.

The study of the stochastic systems related to any nonlinear probabilistic system requires a simulation to the statistical properties for its solution processes and the Wiener Hermite expansion (WHE) method has an interest area related to this study. The WHE method [1-11] introduces analytical treatments for an exited system randomly by the stochastic white noise process and these treatments reduce deterministic system where its solution simulates the statistical moments behavior for the stochastic solution processes.

The differential transform method (DTM) [12] is one of the semi-numerical and analytical methods for ordinary and partial differentialequations that uses the form of polynomials as approximations of the exact solutions that are sufficientlydifferentiable. The DTM has an interest application area in the recent years in the analytical treatments related to nonlinear boundary and initial value problems. The method was developed in different research areas to reach the real behavior under a new concept called the multi-step differential transform method (Ms-DTM) [21-22].

This paper introduces WHE linked by Ms-DTM application to simulate the Gaussian part behavior related to a nonlinear stochastic Van der Poloscillator model. The mathematical description of this problem [23] takes the form,

$$\ddot{x} + \xi(x^2 - 1)\dot{x} + \beta x(t) + \alpha x(t)^3 = \lambda n(t;\omega), \qquad x(0) = a,\ \dot{x}(0) = b \tag{1}$$

where$n(t;\omega)$ is the stochastic white noise process, λ is its intensity parameter and $a, b, \xi, \beta, \alpha$are deterministic values.

This paper is organized as follows. In sections 2-4, we describe a simple survey related to WHE, DTM and Ms-DTM. Sections 5, 6 and 7 describe the application results of WHE , DTM and Ms-DTM respectively. The conclusions are then given in section 8.

II. The stochastic Wiener-Hermite expansion (WHE)

The Wiener–Hermite expansion (WHE) is used to approximate the stochastic processes and this expansion consists of two different quantities, the first is an unknown deterministic and the other is a probabilistic. The probabilistic type includes stochastic processes take the symbolic formula $H^{(i)}(t_1, t_2, \cdots, t_i)$which is called stochastic Wiener – Hermite polynomials (WHPs) and subject to the recurrence relation

$$H^{(i)}(t_1, t_2, \cdots, t_i) = H^{(i-1)}(t_1, t_2, \cdots, t_{i-1})H^{(1)}(t_i) - \sum_{m=1}^{i-1} H^{(i-2)}(t_1, t_2, \cdots, t_{i-2})\delta(t_{i-m} - t_i), \quad i \geq 2, \tag{2}$$

where$H^{(0)} = 1, H^{(1)}(t) = n(t)$ is the stochastic white noise process and $\delta(.)$is the Dirac-delta function. WHPs setare elements of a complete set of statistically orthogonal random functions, i.e.

$$E[\, H^{(i)}(t_1, t_2, \cdots, t_i) H^{(j)}(t_1, t_2, \cdots, t_j)] = 0, \quad \forall i \neq j, \tag{3}$$

where$E[\,.\,]$ denotes the expectation operator.

The completeness of WHPs set, plays an important role to describes the general formula of WHE for any arbitrary stochastic process $v(t;\omega)$ and it can be presented in the form,

$$v(t;\omega) = v^{(0)}(t) + \int_{-\infty}^{\infty} v^{(1)}(t,t_1) H^{(1)}(t_1) dt_1 + \iint_{-\infty}^{\infty} v^{(2)}(t,t_1,t_2) H^{(2)}(t_1,t_2) dt_1 dt_2 + \cdots, \tag{4}$$

where$v^{(0)}(t),\ v^{(i)}(t_1, \ldots, t_i), i \geq 1$are called the (unknown deterministic) kernels of the WHE. The first two terms of the right-hand side (1^{st}order term) define the Gaussian part of the stochastic process, while the second-order and higher terms correspond to the non-Gaussian part.

Under taking expectations linked by the statistical properties of WHPs set (see appendix A), the mean and variance for the Gaussian part of WHE can be expressed as follows:

$$E[v(t;\omega)] = u^{(0)}(t), \quad Var[v(t;\omega)] = \int_{-\infty}^{\infty} \left[v^{(1)}(t,t_1,t_2)\right]^2 dt_1 \tag{5}$$

III. The differential transformation method (DTM)

In this section, we present a review of the DTM. The differential transform of thek^{th} derivative of function $f(t)$is defined as follows,

$$F(k) = \frac{1}{k\,!}\left[\frac{d^k f(t)}{dt^k}\right]_{t=t_0} \quad , \tag{6}$$

where$f(t)$is the original function and $F(k)$is the transformed function and the differential inverse transform of $F(k)$is defined as,

$$f(t) = \sum_{k=0}^{\infty} F(k)(t - t_0)^k. \tag{7}$$

By combining Eqs. (6-7), we get

$$f(t) = \sum_{k=0}^{\infty} \frac{1}{k\,!}\left[\frac{d^k f(t)}{dt^k}\right]_{t=t_0} (t - t_0)^k, \tag{8}$$

which implies that the concept of differential transform is derived from Taylor series expansion, but the method does notevaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which isdescribed by the transformed equations of the original function. For implementation purposes, the function$f(t)$ is expressed by afinite series and Eq. (7)can be written as,

$$f(t) \approx \sum_{k=0}^{N} F(k)(t - t_0)^k, \tag{9}$$

where$N$ is decided by the convergence of natural frequency and the fundamental operations performed by differential transformcan readily be obtained and are listed inappendix B.

IV. The multi-step differential transformation method (Ms-DTM)

Although the DTM is used to provide approximate solutions for a wide class of nonlinear problems in terms of convergentseries with easily computable components, it has some drawbacks: the series solution always converges in a very smallregion and it has slow convergent rate or completely divergent in the wider region [21-22]. To overcome the shortcoming, wepresent in this section the multi-step DTM that we have developed for the numerical solution of differential equations. Forthis purpose, we consider the following nonlinear initial value problem,

$$f\left(t, u, u', \ldots, u^{(p)}\right) = 0, \tag{10}$$

subject to the initial conditions$u^{(k)}(0) = d_k, k = 0,1, \ldots, p-1$.

Let $[0,T]$be the interval over which we want to find the solution of the initial value problem (10). In actual applicationsofthe DTM, the approximate solution of the initial value problem (10)can be expressed by the finite series,

$$u(t) = \sum_{n=0}^{N} a_n t^n, t \in [0,T]. \tag{11}$$

The multi-step approach introduces a new idea for constructing the approximate solution. Assume that the interval $[0,T]$is divided into$M$subintervals$[t_{m-1}, t_m], m = 1,2, \ldots, M$ of equal step size $h = T/M$by using the nodes$t_m = mh$. Themain ideas of the multi-step DTM are as follows. First, we apply the DTM to Eq. (10)over the interval$[0, t_1]$, we will obtainthe following approximate solution,

$$u_1(t) = \sum_{n=0}^{N} a_{1n} t^n, t \in [0, t_1], \tag{12}$$

using the initial conditions $u_1^{(k)}(0) = d_k$. For $m \geq 2$and at each subinterval $[t_{m-1}, t_m]$we will use the initial conditions$u_m^{(k)}(, t_{m-1}) = u_{m-1}^{(k)}(, t_{m-1})$and apply the DTM to Eq. (10)over the interval $[t_{m-1}, t_m]$where t_0 in Eq. (6)is replaced by t_{m-1}. The process is repeated and generates a sequence of approximate solutions $u_m(t), m = 1,2, \ldots, M$ for the solution$u(t)$,

$$u_m(t) = \sum_{n=0}^{N} a_{mn} (t - t_m)^k, t \in [t_m, t_{m+1}], \tag{13}$$

and the final form of $u(t)$ can be written as follow,

$$u(t) = \begin{cases} u_1(t)\,, & t \in [0, t_1] \\ u_2(t), & t \in [t_1 t_2] \\ & \vdots \\ u_M(t) t \in [t_{M-1,} t_M]. \end{cases} \tag{14}$$

V. WHE application to simulate the Gaussian part related to the model

The Gaussian part of the stochastic solution process for the model (1), it can be putted by WHE in the following form

$$x(t;\omega) = x^{(0)}(t) + \int_0^\infty x^{(1)}(t, t_1) H^{(1)}(t_1) dt_1 \quad (15)$$

Substituting from (15) into (1) and taking some expectations linked by WHPs and Dirac delta function properties (see [24] &appendix [A]), a deterministic system is generated in following form

$$L_1\left[x^{(0)}(t)\right] + \frac{1}{3} L_2 \left[\left[x^{(0)}(t)\right]^3 + 3x^{(0)}(t) \int_0^\infty \left[x^{(1)}(t;t_1)\right]^2 dt_1\right] = 0,$$

$$L_1\left[x^{(1)}(t;t_1)\right] + \ L_2 \left[x^{(1)}(t;t_1)\left(\left[x^{(0)}(t)\right]^2 + \int_0^\infty \left[x^{(1)}(t;t_1)\right]^2 dt_1\right)\right] = \lambda\delta(t - t_1),$$

$$x^{(0)}(0) = a, \quad \dot{x}^{(0)}(0) = 0, \qquad x^{(1)}(0, t_1) = 0, \ \dot{x}^{(1)}(0, t_1) = 0, (16)$$

where$L = \left[\frac{d^2}{dt^2} - \xi \frac{d}{dt} + 1\right], L_2 = \left[\xi \frac{d}{dt} + \alpha\right]$

VI. DTM application to simulate the deterministic system approximation

The outputs of DTMapplication linked by the properties of the differential transform (see appendix B), related to the deterministic differential system (16) are described in the following recurrence relations

$$(k + 2)(k + 1)X^{(0)}(k + 2) - \xi(k + 1)X^{(0)}(k + 1) + \ X^{(0)}(k)$$
$$+\frac{\alpha}{3}\sum_{s=0}^{k}\sum_{m=0}^{k-s}\Bigg(X^{(0)}(s)\Big[X^{(0)}(m)X^{(0)}(k - s - m)$$
$$+ 3\int_0^\infty X^{(1)}(m, t_1)\left[X^{(1)}(k - s - m, t_1) + (k - s - m)X^{(1)}(k - s - m + 1, t_1)\right]dt_1\Big]\Bigg) +$$
$$\frac{\xi}{3}(k + 1)\sum_{s=0}^{k+1}\sum_{m=0}^{k-s+1}\left(X^{(0)}(s)\left[X^{(0)}(m)X^{(0)}(k - s - m + 1) + 3\int_0^\infty X^{(1)}(m, t_1)\left[X^{(1)}(k - s - m + 1, t_1) + \right.\right.\right.$$
(k−s−m+1)X1k−s−m+1,t1dt1=0, (17)

$$\int_0^\infty \left[(k + 2)(k + 1)X^{(1)}(k + 2, t_1) - \xi(k + 1)X^{(1)}(k + 1, t_1) + X^{(1)}(k, t_1)\right]dt_1$$
$$+ \alpha\Bigg(\sum_{s=0}^{k}\sum_{m=0}^{k-s}\int_0^\infty X^{(1)}(s, t_1)dt_1\Big[X^{(0)}(m)X^{(0)}(k - s - m)$$
$$+ \int_0^\infty X^{(1)}(m, t_1)X^{(1)}(k - s - m, t_1)dt_1\Big]\Bigg) +$$
$$\xi(k + 1)\left(\sum_{s=0}^{k+1}\sum_{m=0}^{k-s+1}\int_0^\infty X^{(1)}(s, t_1)dt_1\left[X^{(0)}(m)X^{(0)}(k - s - m + 1) + \int_0^\infty X^{(1)}(m, t_1)X^{(1)}(k - s - m + \right.\right.$$
1,t1dt1=λδk,0(18)

where$X^{(0)}(0) = a, \ X^{(0)}(1) = 0, X^{(1)}(0, t_1) = 0, X^{(1)}(1, t_1) = 0$ and the final form for the solutions are described by a finite series, it can be written as follow

$$x^{(0)}(t) = \sum_{k=0}^{N} X^{(0)}(k)\, t^k, \qquad x^{(1)}(t, t_1) = \sum_{k=0}^{N} X^{(1)}(k, t_1)\, t^k (19)$$

VII. Ms-DTM outputs simulation and discussions

The application of DTM reduces a sequence of algebraic equations generated after expanding the recurrence relations (17-18) using a simulated programing by Mathematica 10. The solution of theses algebraic equations determines the coefficients in (19) and the outputs are functions in the initial parameters related the problem. By repeating this process over sequenced steps by a certain range to reach the real behavior of the problem. For every step, a new initial value problem is considered and its conditions are estimated from the obtained solution at final range of the previous step. The mathematical computations related to this method is performed by a symbolic program was designed by Mathematica 10. By another one, a parallel program by the same version uses the NDSolve package to satisfy the result of the previous program.

The application of Ms-DTM includes some cases study simulate the Gaussian solution parameters of the problem and their results are displayed in the figures (1-4) and the tables (1-4) where in tables 1& 3, the values of columns 2 & 3 already indicate to an initial solution for every interval and a final solution for every a previous interval. Also the cases study include results are obtained by NDSolveMathematica software 10 and it is clear that the comparison between Ms-DTM and NDSolve applications gives excellent agreements.

Intervals	$X_0^{(0)}$	$X_1^{(0)}$	$X_2^{(0)}$	$X_3^{(0)}$	$X_4^{(0)}$	$X_5^{(0)}$
$0. \le t \le 0.1$	0	0.3	0.15	0.	−0.01475	−0.007
$0.1 \le t \le 0.2$	0.0314988	0.329938	0.148812	−0.00991628	−0.0451629	−0.1318
$0.2 \le t \le 0.3$	0.0659646	0.359186	0.141313	−0.0481263	−0.168873	−0.372471
$0.3 \le t \le 0.4$	0.100225	0.385125	0.112406	−0.162534	−0.43376	−0.701117
$0.4 \le t \le 0.5$	0.142648	0.400644	0.0299728	−0.414055	−0.833698	−0.806454
$0.5 \le t \le 0.6$	0.182807	0.390479	−0.151093	−0.808993	−1.00181	0.187686
$0.6 \le t \le 0.7$	0.219131	0.331822	−0.450561	−1.14446	−0.412704	2.55837
$0.7 \le t \le 0.8$	0.246647	0.207001	−0.787765	−0.990206	1.26454	3.50899
$0.8 \le t \le 0.9$	0.258641	0.0265551	−0.979827	−0.215778	2.32489	0.193382
$0.9 \le t \le 1.$	0.25182	−0.168427	−0.9135	0.600444	1.51733	−2.86646
$1. \le t \le 1.1$	0.226498	−0.326088	−0.671056	0.919911	0.121745	−2.2506
$1.1 \le t \le 1.2$	0.188029	−0.43364	−0.406701	0.802278	−0.568158	−0.57845
$1.2 \le t \le 1.3$	0.141347	−0.493274	−0.201973	0.552642	−0.610465	0.252812
$1.3 \le t \le 1.4$	0.0904943	−0.519404	−0.0695572	0.342075	−0.430624	0.397494
$1.4 \le t \le 1.5$	0.0381613	−0.524677	0.0110671	0.207609	−0.24954	0.312579
$1.5 \le t \le 1.6$	−0.014	−0.51698	0.0611927	0.135245	−0.120736	0.205844
$1.6 \le t \le 1.7$	−0.0649609	−0.501094	0.0963614	0.104654	−0.0389082	0.127056
$1.7 \le t \le 1.8$	−0.114002	−0.478744	0.126555	0.0999232	0.0108202	0.0781303
$1.8 \le t \le 1.9$	−0.160609	−0.450358	0.157838	0.110498	0.0391418	0.0402508
$1.9 \le t \le 2.$	−0.203851	−0.415295	0.190671	0.129272	0.0525872	0.0148993
$2. \le t \le 2.1$	−0.243009	−0.372465	0.2357	0.151037	0.0544789	−0.00654003
$2.1 \le t \le 2.2$	−0.278042	−0.32056	0.284167	0.171625	0.0483681	−0.0257708
$2.2 \le t \le 2.3$	−0.307083	−0.258428	0.338102	0.186864	0.0287624	−0.044657
$2.3 \le t \le 2.4$	−0.329058	−0.188109	0.396392	0.192257	0.00196332	−0.0643618
$2.4 \le t \le 2.5$	−0.343719	−0.100269	0.452768	0.186753	−0.036844	−0.0858268
$2.5 \le t \le 2.6$	−0.349035	−0.00428862	0.506733	0.163128	−0.0841065	−0.107486
$2.6 \le t \le 2.7$	−0.344253	0.101362	0.549499	0.118068	−0.142823	−0.126358
$2.7 \le t \le 2.8$	−0.328629	0.213969	0.574053	0.0479069	−0.208664	−0.133683
$2.8 \le t \le 2.9$	−0.301368	0.329316	0.574593	−0.0489664	−0.272345	−0.118012
$2.9 \le t \le 3.$	−0.262798	0.44163	0.54263	−0.167579	−0.318009	−0.0592983
$3. \le t \le 3.1$	−0.213376	0.543827	0.4729	−0.297704	−0.324291	0.0406631
$3.1 \le t \le 3.2$	−0.154695	0.626199	0.364841	−0.419298	−0.273069	0.168277
$3.2 \le t \le 3.3$	−0.0895712	0.68758	0.224628	−0.508098	−0.162175	0.270588
$3.3 \le t \le 3.4$	−0.0190885	0.71675	0.0653391	−0.540935	−0.0141061	0.307149
$3.4 \le t \le 3.5$	0.0536976	0.713897	−0.0996845	−0.518918	0.129795	0.254373
$3.5 \le t \le 3.6$	0.122696	0.679509	−0.241591	−0.446117	0.22563	0.130176
$3.6 \le t \le 3.7$	0.188709	0.619794	−0.360901	−0.344931	0.266434	0.0123348
$3.7 \le t \le 3.8$	0.246664	0.537401	−0.448205	−0.240447	0.247986	−0.0778771
$3.8 \le t \le 3.9$	0.295706	0.441489	−0.506373	−0.150751	0.197284	−0.116644
$3.9 \le t \le 4.$	0.33466	0.336433	−0.540943	−0.0836666	0.138435	−0.113198
$4. \le t \le 4.1$	0.362823	0.226232	−0.558911	−0.036934	0.0880095	−0.0860093
$4.1 \le t \le 4.2$	0.379627	0.113613	−0.565954	−0.0110795	0.053702	−0.0507387
$4.2 \le t \le 4.3$	0.385523	0.000279532	−0.566473	0.00651634	0.0371974	−0.0156251
$4.3 \le t \le 4.4$	0.379896	−0.112679	−0.562359	0.0209465	0.0376842	0.0173567
$4.4 \le t \le 4.5$	0.363029	−0.224363	−0.553562	0.0388101	0.0542773	0.0490622
$4.5 \le t \le 4.6$	0.335103	−0.333669	−0.538094	0.0664815	0.089986	0.0833283
$4.6 \le t \le 4.7$	0.29643	−0.438907	−0.512073	0.110148	0.133867	0.106818
$4.7 \le t \le 4.8$	0.247644	−0.537428	−0.469888	0.174895	0.180645	0.11684
$4.8 \le t \le 4.9$	0.189297	−0.625338	−0.404856	0.262173	0.243367	0.0868753
$4.9 \le t \le 5.$	0.122003	−0.697427	−0.310899	0.366964	0.268034	0.00180434

Table 1: The series coefficients for piecewise solutions by Ms-DTM for $E[x(t)]$ at$\lambda = 1, \beta = 1, \alpha = 3,\ \xi = 1,\ a = 0,\ b = 0.3$

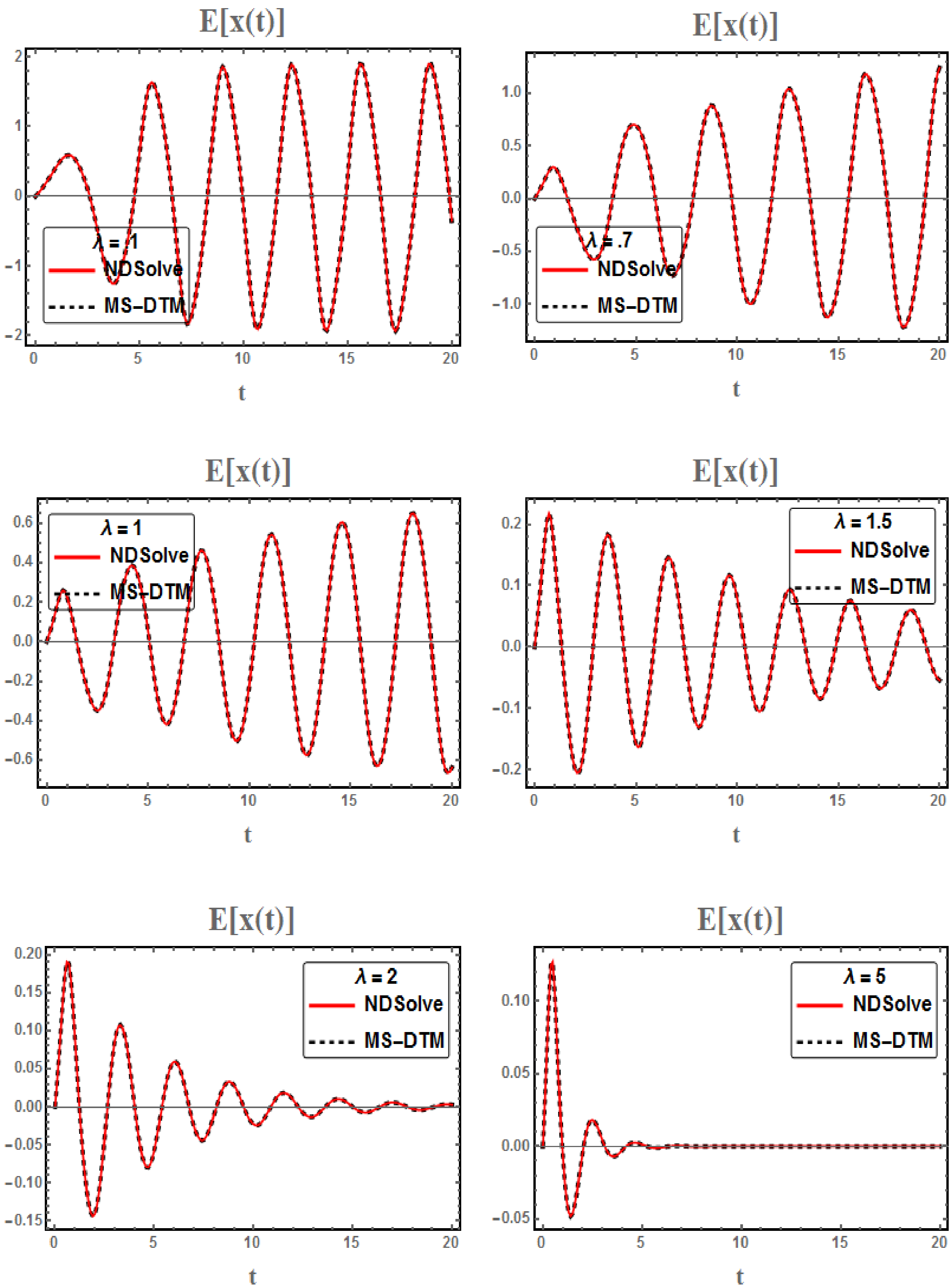

Fig.1: Variations of $E[x(t)]$ with t by Ms-DTM and NDSolveMathematica Software for different values of λ at $\beta = 1, \alpha = 3,\ \xi = 1,\ a = 0,\ b = .3$

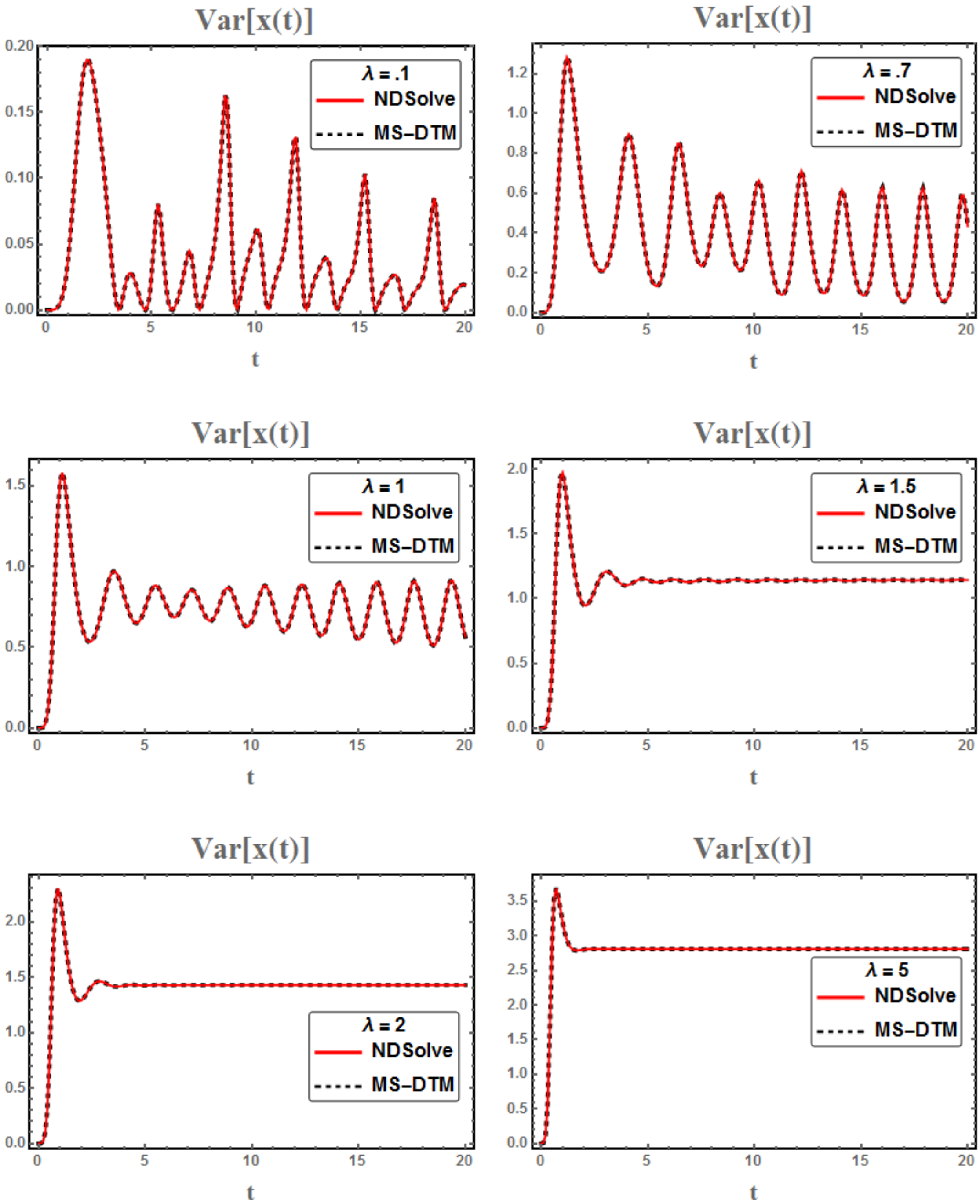

Fig.2: Variations of $Var[x(t)]$ with t by Ms-DTM and NDSolve Mathematica Software for different values of λ at $\beta = 1, \alpha = 3,\ \xi = 1,\ a = 0, b = .3$

t	E[x(t)] Ms−DTM	E[x(t)] NDSolve	Var[x(t)] Ms−DTM	Var[x(t)] NDSolve
0.	0	0	0.	0.
0.5	0.182487	0.182507	0.142431	0.142455
1.	0.226472	0.226468	0.391249	0.391231
1.5	−0.0140197	−0.014	0.339601	0.339608
2.	−0.2433	−0.243309	0.253461	0.253445
2.5	−0.348987	−0.349035	0.232686	0.232674
3.	−0.213317	−0.213376	0.275177	0.275176
3.5	0.123626	0.123596	0.311584	0.311588
4.	0.362792	0.362823	0.286478	0.286482
4.5	0.33503	0.335102	0.255479	0.255471
5.	0.0504739	0.0505431	0.273636	0.273626
5.5	−0.303472	−0.303455	0.297144	0.297146
6.	−0.414542	−0.414598	0.273654	0.273661
6.5	−0.211329	−0.211409	0.264056	0.264048
7.	0.18397	0.183917	0.289379	0.28937
7.5	0.447557	0.447585	0.281282	0.281289
8.	0.370673	0.370754	0.257741	0.25774
8.5	−0.00119753	−0.00112085	0.278581	0.278571
9.	−0.402888	−0.402873	0.29254	0.29254
9.5	−0.489059	−0.489127	0.259183	0.259189
10.	−0.217142	−0.217235	0.258102	0.258093
10.5	0.260139	0.260077	0.295123	0.295113
11.	0.534774	0.534811	0.274979	0.27499
11.5	0.414789	0.41488	0.244446	0.244442
12.	−0.0297925	−0.0297059	0.277276	0.277262
12.5	−0.482217	−0.4822	0.29399	0.293994
13.	−0.551702	−0.551775	0.249165	0.249173
13.5	−0.231119	−0.231214	0.249743	0.249731
14.	0.311135	0.311068	0.297695	0.297682
14.5	0.600692	0.600734	0.270702	0.270716
15.	0.455167	0.455255	0.233981	0.233978
15.5	−0.0396307	−0.0395507	0.274899	0.274883
16.	−0.535194	−0.535181	0.295259	0.295264
16.5	−0.60097	−0.601041	0.241757	0.241766
17.	−0.254287	−0.25437	0.241836	0.241824
17.5	0.334857	0.334798	0.299216	0.299199
18.	0.644869	0.644913	0.268994	0.269009
18.5	0.491906	0.491981	0.226096	0.226093
19.	−0.0295156	−0.0294588	0.271044	0.271029
19.5	−0.562166	−0.562161	0.297245	0.297248
20.	−0.636857	−0.63692	0.238056	0.238065

Tabl.2: Comparison between some results obtained by Ms-DTM and NDSolveMathematica software for $E[x(t)]$ and $Var[x(t)]$at$\lambda = 1,\ \beta = 1, \alpha = 3,\ \xi = 1,\ a = 0,\ b = .3$

Intervals	$X_0^{(0)}$	$X_1^{(0)}$	$X_2^{(0)}$	$X_3^{(0)}$	$X_4^{(0)}$	$X_5^{(0)}$
$0. \le t \le 0.1$	0	0.3	0.45	0.4	0.25575	0.1087
$0.1 \le t \le 0.2$	0.0349267	0.403077	0.586096	0.507838	0.261581	−0.162137
$0.2 \le t \le 0.3$	0.0816277	0.536497	0.750758	0.569622	−0.0461145	−1.29295
$0.3 \le t \le 0.4$	0.143337	0.702903	0.900386	0.340276	−1.35114	−4.28924
$0.4 \le t \le 0.5$	0.222793	0.88564	0.868877	−0.760962	−4.42823	−7.33465
$0.5 \le t \le 0.6$	0.318769	1.01521	0.311235	−3.0964	−6.36085	3.54536
$0.6 \le t \le 0.7$	0.419705	0.96089	−0.898858	−4.43254	1.45904	24.4641
$0.7 \le t \le 0.8$	0.502763	0.666211	−1.91862	−1.78948	9.73276	1.4233
$0.8 \le t \le 0.9$	0.549396	0.268444	−1.94159	1.32262	4.45673	−15.839
$0.9 \le t \le 1.$	0.558435	−0.0702877	−1.41268	1.81748	−1.00625	−4.98615
$1. \le t \le 1.1$	0.538946	−0.304818	−0.952498	1.18448	−1.60776	1.10691
$1.1 \le t \le 1.2$	0.499974	−0.465661	−0.67885	0.686976	−0.841086	1.51555
$1.2 \le t \le 1.3$	0.447238	−0.583428	−0.509813	0.480756	−0.25193	0.82848
$1.3 \le t \le 1.4$	0.384261	−0.671561	−0.374147	0.442273	0.00849005	0.256043
$1.4 \le t \le 1.5$	0.313809	−0.732961	−0.2395	0.457899	0.0385858	−0.0974803
$1.5 \le t \le 1.6$	0.238578	−0.767018	−0.101341	0.457046	−0.0560625	−0.246402
$1.6 \le t \le 1.7$	0.161312	−0.773922	0.0298891	0.409752	−0.178426	−0.21688
$1.7 \le t \le 1.8$	0.0846086	−0.756474	0.140226	0.320693	−0.256113	−0.084428
$1.8 \le t \le 1.9$	0.0106677	−0.719875	0.220594	0.214673	−0.262748	0.0520259
$1.9 \le t \le 2.$	−0.0589349	−0.67034	0.270004	0.117972	−0.214279	0.130298
$2. \le t \le 2.1$	−0.123171	−0.613592	0.293919	0.0462233	−0.143306	0.144599
$2.1 \le t \le 2.2$	−0.181558	−0.553923	0.300588	0.00276058	−0.0760683	0.120368
$2.2 \le t \le 2.3$	−0.233948	−0.493967	0.297963	−0.0168058	−0.0248119	0.0843341
$2.3 \le t \le 2.4$	−0.280383	−0.434935	0.292185	−0.019435	0.0089241	0.0518724
$2.4 \le t \le 2.5$	−0.320973	−0.37702	0.28734	−0.0115462	0.0285032	0.0279253
$2.5 \le t \le 2.6$	−0.35581	−0.31977	0.285819	0.00205624	0.0381603	0.0118289
$2.6 \le t \le 2.7$	−0.384923	−0.262386	0.288814	0.0181102	0.0412126	0.0010341
$2.7 \le t \le 2.8$	−0.408251	−0.203914	0.296709	0.034414	0.039637	−0.00709647
$2.8 \le t \le 2.9$	−0.425637	−0.143385	0.309323	0.0493062	0.0341992	−0.0147398
$2.9 \le t \le 3.$	−0.43683	−0.0799119	0.326	0.0612309	0.0247014	−0.0235819
$3. \le t \le 3.1$	−0.441498	−0.0127881	0.34559	0.0684	0.0102199	−0.0348293
$3.1 \le t \le 3.2$	−0.439251	0.0584055	0.366343	0.0685555	−0.0106087	−0.0489568
$3.2 \le t \le 3.3$	−0.42968	0.133664	0.385746	0.0588877	−0.0390625	−0.0649794
$3.3 \le t \le 3.4$	−0.412402	0.212391	0.40038	0.0362519	−0.0753156	−0.0792383
$3.4 \le t \le 3.5$	−0.387131	0.293214	0.40592	−0.00207891	−0.116791	−0.084275
$3.5 \le t \le 3.6$	−0.353765	0.373826	0.397462	−0.0569308	−0.156287	−0.0694196
$3.6 \le t \le 3.7$	−0.312481	0.450951	0.370394	−0.125172	−0.181323	−0.0254423
$3.7 \le t \le 3.8$	−0.263826	0.520536	0.321866	−0.198025	−0.177027	0.0462548
$3.8 \le t \le 3.9$	−0.208769	0.578284	0.2525	−0.261493	−0.13373	0.125579
$3.9 \le t \le 4.$	−0.148689	0.620467	0.167447	−0.300311	−0.0559667	0.178048
$4. \le t \le 4.1$	−0.0852722	0.644812	0.0758175	−0.304469	0.0350335	0.175737
$4.1 \le t \le 4.2$	−0.020332	0.65107	−0.0117712	−0.274407	0.110623	0.119295
$4.2 \le t \le 4.3$	0.0443961	0.640985	−0.0864623	−0.220889	0.15023	0.0380586
$4.3 \le t \le 4.4$	0.107423	0.617686	−0.143531	−0.159547	0.15062	−0.0324911
$4.4 \le t \le 4.5$	0.167612	0.58478	−0.182809	−0.104166	0.122944	−0.0726963
$4.5 \le t \le 4.6$	0.224169	0.545548	−0.207453	−0.0628136	0.082988	−0.0828956
$4.6 \le t \le 4.7$	0.276594	0.502464	−0.222132	−0.0377264	0.0432064	−0.0741147
$4.7 \le t \le 4.8$	0.324686	0.457041	−0.231568	−0.0273441	0.0100848	−0.0578938
$4.8 \le t \le 4.9$	0.367947	0.409921	−0.239691	−0.0285422	−0.014693	−0.0415696
$4.9 \le t \le 5.$	0.406611	0.361047	−0.249513	−0.0381031	−0.0319781	−0.0281406

Table 3: The series coefficients for piecewise solution by Ms-DTM for $E[x(t)]$ at$\lambda = 0.6, \beta = 1, \alpha = 1,\ \xi = 3,\quad a = 0,\ b = 0.3$

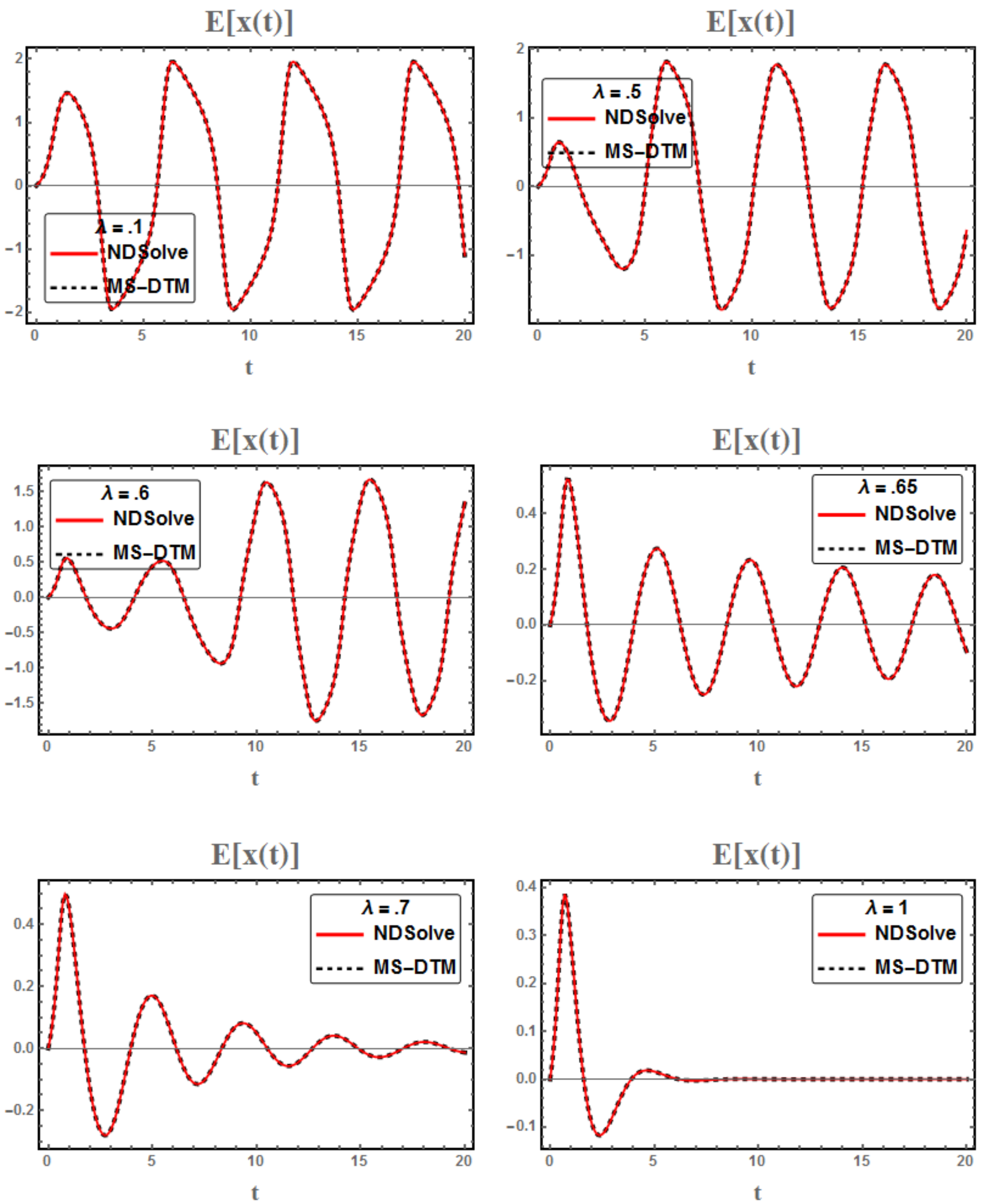

Fig.3: Variations of $E[x(t)]$ with t by Ms-DTM and NDSolveMathematicasoftware for different values of λ at $\beta = 1,\ \alpha = 1,\ \xi = 3,\ a = 0,\ b = 0.3$

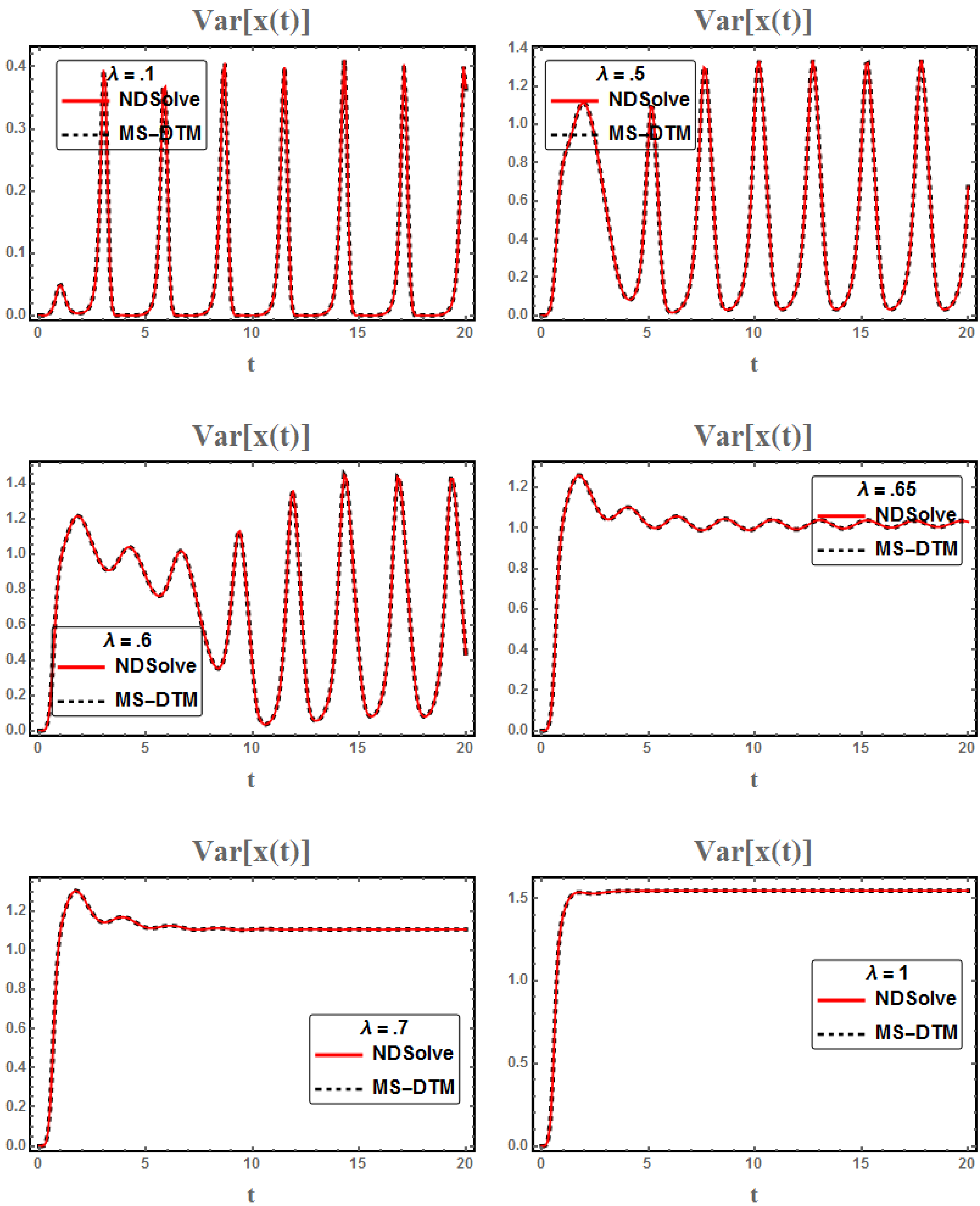

Fig.4: Variations of $Var[x(t)]$ with t by Ms-DTM and NDSolveMathematicasoftware for different values of λ at $\beta = 1,\ \alpha = 1,\ \xi = 3,\ a = 0,\ b = 0.3$

t	E[x(t)] Ms−DTM	E[x(t)] NDSolve	Var[x(t)] Ms−DTM	Var[x(t)] NDSolve
0.	0	0	0.	0.
0.5	0.325064	0.325151	0.0823878	0.0824282
1.	0.771285	0.771347	0.245418	0.245449
1.5	0.681692	0.681686	0.257625	0.257623
2.	0.209784	0.209749	0.302949	0.302949
2.5	−0.462565	−0.462567	0.297656	0.29766
3.	−1.1539	−1.154	0.161472	0.161445
3.5	−1.58958	−1.58963	0.0119671	0.011955
4.	−1.38435	−1.38435	0.0240656	0.024064
4.5	−0.934909	−0.934921	0.107057	0.107056
5.	0.324174	0.324195	0.321707	0.321528
5.5	1.64153	1.64164	0.127586	0.127532
6.	1.82972	1.82976	0.0290305	0.029011
6.5	1.54924	1.54922	0.0589185	0.0589214
7.	1.1125	1.1125	0.134558	0.134559
7.5	−0.0033734	−0.00367659	0.329993	0.329949
8.	−1.27546	−1.27566	0.221562	0.22151
8.5	−1.84012	−1.84028	0.0362127	0.0361637
9.	−1.61285	−1.61283	0.0503114	0.0503127
9.5	−1.22854	−1.22853	0.111302	0.111302
10.	−0.319352	−0.318963	0.287866	0.287938
10.5	1.03776	1.0379	0.274345	0.274355
11.	1.83762	1.83765	0.0475563	0.0475551
11.5	1.67435	1.67434	0.0430917	0.0430915
12.	1.32732	1.32731	0.0932461	0.093247
12.5	0.603217	0.603072	0.237543	0.237592
13.	−0.795886	−0.796005	0.31571	0.315785
13.5	−1.78244	−1.7822	0.0732049	0.0732791
14.	−1.73091	−1.73091	0.0374764	0.0374715
14.5	−1.4136	−1.41358	0.0787326	0.0787338
15.	−0.828383	−0.828363	0.192319	0.192332
15.5	0.539569	0.539496	0.341592	0.34162
16.	1.65386	1.65383	0.117091	0.117081
16.5	1.78112	1.78115	0.033794	0.0337772
17.	1.49093	1.49092	0.0667962	0.0667976
17.5	1.00223	1.00224	0.156296	0.156297
18.	−0.251254	−0.251162	0.34558	0.345459
18.5	−1.45688	−1.4571	0.174629	0.174555
19.	−1.82129	−1.82139	0.0330891	0.033051
19.5	−1.56158	−1.56156	0.0568489	0.0568503
20.	−1.13936	−1.13936	0.128501	0.1285

Table 4: Comparison between some results obtained byMs-DTM and NDSolveMathematica software for$E[x(t)]$ and $Var[x(t)]$ at$\lambda = 0.65,\ \beta = 1,\ \alpha = 1,\ \xi = 3,\ a = 0,\ b = 0.3$

VIII. Conclusions

In this paper, the Wiener Hermite expansion technique was applied to simulate the Gaussian part parameters (mean and variance) of the stochastic solution processes related to a stochastic model describes

Vander Pol–Duffingoscillator model. Due to apply this method, a deterministic nonlinear system was generated its solution describes the Gaussian part parameters. The approximations of the deterministic system were obtained by the concepts of multi-step differential transformed method and its results were compared with NDSolveMathematicasoftware package. Some cases study were introduced to illustrate the results of analysis.

Appendix A

The statistical properties of Wiener Hermite polynomials (WHPs) [10-11] which were used in this paper are simulated in the following items

- $E[\,H^{(1)}(t_1)H^{(1)}(t_2)] = \delta(t_2 - t_1)$
- $E[\,H^{(1)}(t_1)H^{(1)}(t_2)H^{(1)}(t_3)] = 0$
- $E[\,H^{(1)}(t_1)H^{(1)}(t_2)H^{(1)}(t_3)H^{(1)}(t_4)] = \delta(t_2 - t_1)\delta(t_3 - t_4) + \delta(t_1 - t_3)\delta(t_2 - t_4) + \delta(t_1 - t_4)\delta(t_2 - t3$

Appendix B

In this paper, the used properties [22] related to the differential transformation$F(k)$ for a function $f(t)$ are stated in the following items

- $f(t) = mu(t) \pm nv(t) \Rightarrow F(k) = mU(k) \pm nV(k)$
- $f(t) = u(t)v(t) \Rightarrow F(k) = \sum_{l=0}^{k} U(l)\,V(k-l)$
- $f(t) = u(t)v(t)w(t) \qquad \Rightarrow F(k) = \sum_{l=0}^{k}\sum_{s=0}^{k-l} U(l)\,V(s)\,W(k-l-s)$
- $f(t) = \frac{d^m u(t)}{dt^m} \Rightarrow F(k) = \frac{(k+m)!}{k!}U(k+m)$
- $f(t) = t^m \Rightarrow F(k) = \delta_{k,m}$ where $\delta_{k,m}$is Kronecker's delta

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